Tilting for commutative rings and its applications
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part I: tilting modules and approximations

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Overview of the lecture series topics

I. Tilting modules and classes.

II. Structure of tilting and cotilting classes over commutative noetherian rings.

III. Zariski locality for tilting quasi-coherent sheaves.

Overview of this talk

I. Tilting modules and classes.

II. Tilting classes are of finite type.

III. An application to Iwanaga-Gorenstein rings.

IV. Tilting approximations.

I. Tilting modules and classes

Tilting modules

Let R be a ring and $n < \omega$. A right R-module T is n–tilting provided

(T1) $\mathrm{pd}_R(T) \leq n$, *i.e., there is an* $\mathrm{Add}(R)$*-resolution of* T *of length* $\leq n$.

(T2) $\mathrm{Ext}^i_R(T, T^{(\kappa)}) = 0$ for all $1 \leq i$ and all κ, *i.e.,* T *is a strong splitter.*

(T3) There is a long exact sequence $0 \to R \to T_0 \to \cdots \to T_n \to 0$ with $T_i \in \mathrm{Add}\, T$, *i.e., there is an* $\mathrm{Add}(T)$*-coresolution of* R *of length* $\leq n$.

$\mathrm{Add}(T)$ = all direct summands (of possibly infinite) direct sums of copies of T.

Tilting module = n–tilting module for some $n < \omega$.

Tilting modules

A tilting module T is

- good if (T3) holds with $\operatorname{Add} T$ replaced by $\operatorname{add} T$.
- classical if $T \in \mathrm{mod}\text{–}R$ (i.e., T is strongly finitely presented).

Example

T is a 0-tilting module, iff T is a projective generator. Such T is always good, and it is classical, iff T is finitely generated.

Two tilting modules T and T' are equivalent if $\operatorname{Add} T = \operatorname{Add} T'$.

Lemma

- Each tilting module is equivalent to a good one.
- Each classical tilting module is good.

Tilting classes

For a tilting module T, we define

$\mathcal{B} = T^{\perp} = \bigcap_{i>0} \operatorname{KerExt}^i_R(T, -)$ is the (right) tilting class of T.

$\mathcal{A} = \operatorname{KerExt}^1_R(-, \mathcal{B})$ is the left tilting class of T.

Lemma

- $\mathcal{A}$ is the class of all modules possessing an $\operatorname{Add} T$-coresolution of length $\leq n$.
- $\mathcal{B}$ is the class of all modules possessing an $\operatorname{Add} T$-resolution.
- $\mathcal{A} \cap \mathcal{B} = \operatorname{Add} T$.

Tilting theorems

The classic case - Miyashita'1986

Let T be a classical n–tilting module. Then for each $i \leq n$ there is a category equivalence

$$\bigcap_{j \leq n, j \neq i} \mathrm{Ker}(\mathrm{Ext}^j_R(T,-)) \underset{\mathrm{Tor}^i_S(-,T)}{\overset{\mathrm{Ext}^i_R(T,-)}{\rightleftarrows}} \bigcap_{j \leq n, j \neq i} \mathrm{Ker}(\mathrm{Tor}^S_j(-,T))$$

where $S = \mathrm{End}\, T$.

Tilting theorems

The good case - Bazzoni'2011

Let R be a ring and T be a good n–tilting module. Then for each $i \leq n$ there is a category equivalence

$$\bigcap_{j \leq n, j \neq i} \mathrm{Ker}(\mathrm{Ext}^j_R(T,-)) \underset{\mathrm{Tor}^i_S(-,T)}{\overset{\mathrm{Ext}^i_R(T,-)}{\rightleftarrows}} \bigcap_{j \leq n, j \neq i} \mathrm{Ker}(\mathrm{Tor}^S_j(-,T)) \cap \mathcal{E}_\perp$$

where $S = \mathrm{End}\, T$, $\mathcal{E}_\perp = \{X \in \mathcal{D}(S) \mid \mathrm{Hom}_{\mathcal{D}(S)}(\mathcal{E}, X) = 0\}$, and $\mathcal{E}$ is the kernel of the total left derived functor $\mathbb{L}(- \otimes_S T)$.

A basic restriction for the commutative setting

Lemma

(i) If $T \in \mathrm{mod}\text{–}R$ and $1 \leq \mathrm{pd}_R(T) = n < \infty$, then $\mathrm{Ext}^n_R(T, T) \neq 0$. Hence T fails condition (T2).

(ii) Each classical tilting module is projective.

Proof

(i) All syzygies of T are finitely presented, so $\mathrm{pd}_R(T) = \max_m \mathrm{pd}_{R_m}(T_m)$. Take $m \in \mathrm{mSpec}(R)$ such that $\mathrm{pd}_{R_m}(T_m) = n$. Then $T_m \in \mathrm{mod}\text{–}R_m$ and $(\mathrm{Ext}^n_R(T, T))_m \cong \mathrm{Ext}^n_{R_m}(T_m, T_m)$, so w.l.o.g., we can assume that R is local. Then T has a minimal free resolution, $\mathcal{F}$, given by an iteration of projective covers. So $d_k(F_k) \subseteq mF_{k-1}$ for all $k > 0$ where d_k is the differential. As $\mathrm{Ext}^n_R(T, -)$ is right exact, the epimorphism $T \to T/mT$ induces a surjection $\mathrm{Ext}^n_R(T, T) \to \mathrm{Ext}^n_R(T, T/mT)$. However, $\mathrm{Ext}^n_R(T, T/mT) \cong \mathrm{Ext}^1_R(\Omega^{n-1}(T), T/mT) \cong \mathrm{Hom}_R(F_n, T/mT) \neq 0$.
(ii) By part (i). □

Example: tilting over Dedekind domains

Let R be a Dedekind domain with the quotient field Q.

For each $P \subseteq \mathrm{mSpec}(R)$, let A_P be the module defined by $R \subseteq A_P \subseteq Q$ and $A_P/R \cong \bigoplus_{p \notin P} E(R/p)$.

Example

$T_P = A_P \oplus A_P/R$ is a tilting module with the tilting class $\mathcal{B}_P = \{A \in \mathrm{Mod}\text{–}R \mid Ap = A \text{ for all } p \notin P\}$.

Theorem

Each tilting module T is equivalent to T_P for some $P \subseteq \mathrm{mSpec}(R)$.

II. Tilting classes are of finite type

The finite type theorem

Theorem

Let T be a tilting module with the left and right tilting classes $\mathcal{A}$ and $\mathcal{B}$.

Then there is a subset $\mathcal{S} \subseteq \mathcal{A}$ consisting of strongly finitely presented modules, such that

$$\mathcal{B} = \mathrm{KerExt}^1_R(\mathcal{S}, -).$$

The largest choice for $\mathcal{S}$ is $\mathcal{S} = \mathcal{A} \cap \mathrm{mod}\text{–}R$; in this case $\mathcal{A} \subseteq \varinjlim \mathcal{S}$.

Structure of the proof: First, one proves countable type by set-theoretic homological algebra, by induction on $\mathrm{pd}_R(T)$.
Then one proceeds to finite type using Mittag-Leffler conditions.

Consequences of the finite type

Relations to logic

Each right tilting class $\mathcal{B}$ is definable, i.e., closed under direct limits, pure submodules and products.

In fact, elements of $\mathcal{B}$ can be characterized by a set of first order formulas of the language of the theory of modules.

Resolving subcategories

A subcategory $\mathcal{C}$ of mod–R is called resolving, if $\mathcal{C}$ contains all finitely generated projective modules, $\mathcal{C}$ is closed under extensions and direct summands, and $A \in \mathcal{C}$ whenever A fits in a short exact sequence $0 \to A \to B \to C \to 0$ with $B, C \in \mathcal{C}$.

Consequences of the finite type

Corollary

n-tilting classes $\mathcal{B}$ correspond 1-1 to resolving subcategories $\mathcal{C}$ of $\mathrm{mod}\text{–}R$ consisting of modules of projective dimension $\leq n$.

The correspondence: $\mathcal{B} \mapsto \mathrm{mod}\text{–}R \cap {}^{\perp}\mathcal{B}$, and $\mathcal{C} \mapsto \mathcal{C}^{\perp}$.

Dedekind domains revisited

For $P \subseteq \mathrm{mSpec}(R)$, we can take $\mathcal{S}_P = \{R/p \mid p \in P\}$.

III. An application to Iwanaga-Gorenstein rings

Finitistic dimensions of Iwanaga-Gorenstein rings

- Let $n \geq 0$. A ring R is n-Iwanaga-Gorenstein, if R is left and right Noetherian, and both the left and the right injective dimension of R equals n. E.g., 0-Gorenstein = quasi-Frobenius.
- The big finitistic dimension of a ring R, denoted by Fdim(R), is the supremum of projective dimensions of all modules that have finite projective dimension.
- The little finitistic dimension, denoted by fdim(R), is the supremum of projective dimensions of all finitely generated modules that have finite projective dimension.

Theorem

Let R be an n-Iwanaga-Gorenstein ring. Then Fdim(R) = fdim(R) = n.

Proof

Since R is n-Iwanaga-Gorenstein, $\mathcal{P} = \mathcal{P}_n = \mathcal{I}_n$, whence Fdim(R) = n. Let $\mathcal{S} = \mathcal{P} \cap \mathrm{mod}\text{–}R$.

Idea of the proof: Show that $\mathcal{P}$ is the left tilting class of an (infinitely generated) n-tilting module T. Then the finite type of T will give $\mathcal{P}^\perp = \mathcal{S}^\perp$, whence $\mathcal{P} = {}^\perp(\mathcal{S}^\perp)$, and Fdim(R) = fdim(R).

Constructing the tilting module T: Consider the minimal injective coresolution of R, $0 \to R \to I_0 \to \cdots \to I_n \to 0$. Then $T = \bigoplus_{i \leq n} I_i$ is obviously an n-tilting module. Let $\mathcal{A}$ and $\mathcal{B}$ be the left and right tilting classes of T. Then $\mathcal{A} \cap \mathcal{B} = \mathrm{Add}\, T \subseteq \mathcal{I}_0$.

We will show that $\mathrm{Add}\, T = \mathcal{I}_0$: Let I be injective and $B \in \mathcal{B}$. Then B has an $\mathrm{Add}\, T$-resolution $\mathcal{R}$. Let B_n be the n-the syzygy of B in $\mathcal{R}$. Then $\mathrm{Ext}^1_R(I, B) \cong \mathrm{Ext}^{n+1}_R(I, B_n) = 0$, because $\mathrm{Add}\, T \subseteq \mathcal{I}_0$ and $I \in \mathcal{P}_n$. So $I \in \mathcal{A}$, and $I \in \mathrm{Add}\, T$.

$\mathcal{A}$ = the class of all modules possessing a $\mathcal{I}_0$-coresolution of length $\leq n$. That is, $\mathcal{A} = \mathcal{I}_n = \mathcal{P}$. So $\mathcal{P}$ is the left tilting class of T. □

The tilting module T_f behind the scenes

There is a partial order $\preceq$ on (the equivalence classes of) tilting modules by inclusion of the left tilting classes.

Assume that R is a right noetherian ring. Then $\mathrm{fdim}(R) < \infty$, iff there exists a $\preceq$-greatest element T_f. In this case $\mathrm{fdim}(R) = \mathrm{pd}_R(T_f)$.

Example

In the Iwanaga-Gorenstein case above, $T_f = \bigoplus_{i \leq n} I_i$.

The Finitistic Dimension Conjecture

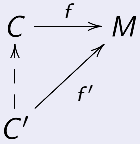
Does $\mathrm{fdim}(R) < \infty$ hold for each artin algebra R?

Note: Even if R is an artin algebra, T_f need not be classical. (The latter happens iff $\mathcal{P} \cap \mathrm{mod}\text{–}R$ is a covering class in $\mathrm{mod}\text{–}R$.)

IV. Tilting approximations

Approximations of modules

A class of modules $\mathcal{C}$ is precovering if for each module M there is $f \in \mathrm{Hom}_R(C, M)$ with $C \in \mathcal{C}$ such that each $f' \in \mathrm{Hom}_R(C', M)$ with $C' \in \mathcal{C}$ has a factorization through f:



The map f is called an $\mathcal{C}$–precover of M. If moreover f is surjective and $\mathrm{Ext}^1_R(C, \mathrm{Ker}(f)) = 0$ for each $C \in \mathcal{C}$, then f is a special $\mathcal{C}$–precover of M.

If moreover f is right minimal (i.e., $fg = f$ implies g is an automorphism for each $g \in \mathrm{End}\,C$), then f is a $\mathcal{C}$–cover of M.

If each module has a special $\mathcal{A}$–precover ($\mathcal{A}$-cover), then $\mathcal{A}$ is special precovering (covering).

Approximations and tilting

The notions a (special) $\mathcal{C}$–preenvelope, $\mathcal{C}$–envelope, and of a (special) preenveloping and enveloping class are defined dually.

1-dimensional case

TFAE for a class of modules $\mathcal{C}$:

- $\mathcal{C}$ is a right tilting class of some 1-tilting module.
- $\mathcal{C}$ is a special preenveloping torsion class of modules.

n-dimensional case

Let n be a natural number. TFAE for a class of modules $\mathcal{C}$:

- $\mathcal{C}$ is a (right) tilting class of some n-tilting module.
- $\mathcal{C}$ is a special preenveloping class closed under direct sums and direct summands, $\mathcal{C}$ is coresolving, and ${}^{\perp}\mathcal{C}$ consists of modules of projective dimension $\leq n$.

Reference

R. Goebel, J. Trlifaj,, *Approximations and endomorphism algebras of modules*, 2nd rev. & ext. ed., GEM 41, W. de Gruyter, Berlin-Boston 2012.